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A quandle cocycle invariant with non-commutative flows for a handlebody-knot

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Abstract

This is a summary of the construction of the quandle cocycle invariant obtained in the joint work with Iwakiri, Jang and Oshiro [7]. Iwakiri and the author [6] introduced a notion of a flow, and defined a quandle cocycle invariant for handlebody-knots. The quandle cocycle invariant given in this article is defined by using "non-commutative" flows.

1 A G-family of quandles

A quandle [8, 9] is a non-empty set X with a binary operation $*: X\times X\to X$ satisfying

- $x*x=x \quad (x\in X)$,
- $*x: X\to X$ is bijective $\quad (x\in X)$,
- $(x*y)*z=(x*z)*(y*z) \quad (x,y,z\in X)$.

An Alexander quandle $(M,*)$ is a Λ-module M with the binary operation defined by $x*y=tx+(1-t)y$, where $\Lambda:=\mathbb{Z}[t,t^{-1}]$. A conjugation quandle $(G,*)$ is a group G with the binary operation defined by $x*y=y^{-1}xy$.

A G-family of quandles is a non-empty set X with a family of binary operations $*^{g}: X\times X\to X$ $(g\in G)$ satisfying

- $x*^{g}x=x \quad (x\in X,\ g\in G)$,
- $x*^{gh}y=(x*^{g}y)*^{h}y,\ x*^{e}y=x \quad (x,y\in X,\ g,h\in G)$,

- $(x *^g y) *^h z = (x *^h z) *^{h^{-1}gh} (y *^h z) \quad (x, y, z \in X,\ g, h \in G)$.

Proposition 1. Let G be a group, and $(X, \{*^g\}_{g \in G})$ a G-family of quandles.

(1) For any $g \in G$, $(X, *^g)$ is a quandle.

(2) We define $* : (X \times G) \times (X \times G) \to X \times G$ by

$$(x, g) * (y, h) = (x *^h y, h^{-1}gh).$$

Then $(X \times G, *)$ is a quandle

We call the quandle $(X \times G, *)$ given in Proposition 1 the *associated quandle* of X.

Proposition 2. Let R be a ring, G a group, and X a right $R[G]$-module. We define a binary operation $*^g : X \times X \to X$ by $x *^g y = xg + y(e - g)$. Then X is a G-family of quandles.

Let X be a G-family of quandles, and Q the associated quandle of X. The *associated group* of X, denoted by $\mathrm{As}(X)$, is defined by

$$\mathrm{As}(X) = \left\langle q \in Q \;\middle|\; \begin{array}{l} q_1 * q_2 = q_2^{-1} q_1 q_2 \ (q_1, q_2 \in Q), \\ (x, gh) = (x, g)(x, h) \ (x \in X,\ g, h \in G) \end{array} \right\rangle.$$

An X*-set* Y is a set equipped with a right action of the associated group $\mathrm{As}(X)$. We denote by $y * q$ the image of an element $y \in Y$ by the action $q \in \mathrm{As}(X)$. We also denote $y * (x, g)$ by $y *^g x$. Any singleton set $\{y\}$ is an X-set with the trivial action, which is a trivial X-set. The set X is also an X-set with the action defined by $y * (x, g) = y *^g x$ for $y \in X$, $(x, g) \in Q$.

2 A handlebody-link

A *handlebody-link* is a disjoint union of handlebodies embedded in the 3-sphere S^3. Two handlebody-links are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other. A *spatial graph* is a finite graph embedded in S^3. Two spatial graphs are *equivalent* if there is an orientation-preserving self-homeomorphism of S^3 which sends one to the other. When a handlebody-link H is a regular neighborhood of a spatial graph K, we say that H is represented by K. In this article,

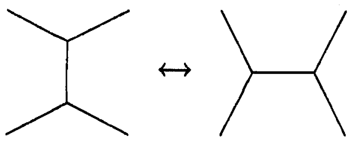

Figure 1:

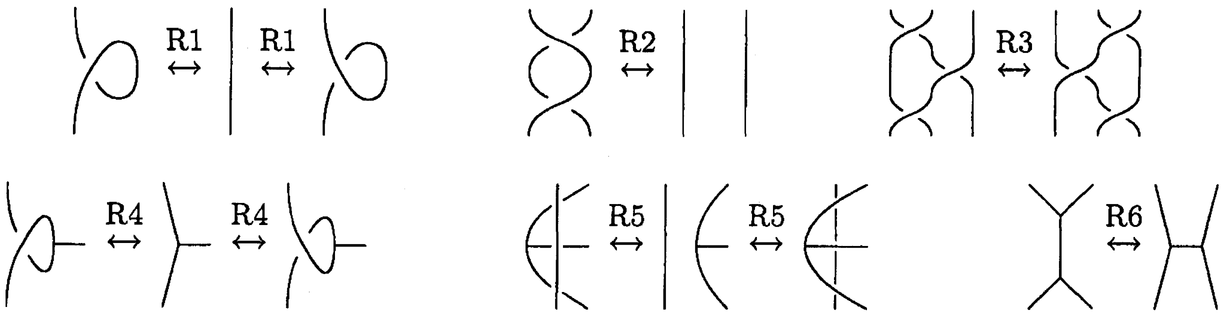

Figure 2:

a trivalent graph may contain circle components. Then any handlebody-link can be represented by some spatial trivalent graph. A *diagram* of a handlebody-link is a diagram of a spatial trivalent graph which represents the handlebody-link. An *IH-move* is a local spatial move on spatial trivalent graphs as described in Figure 1.

Theorem 3 ([5]). For spatial trivalent graphs K_1 and K_2, the following are equivalent.

- K_1 and K_2 represent an equivalent handlebody-link.
- K_1 and K_2 are related by a finite sequence of IH-moves.
- Diagrams of K_1 and K_2 are related by a finite sequence of the moves depicted in Figure 2.

3 A coloring with G-family of quandles

Let D be a diagram of a handlebody-link H. Putting an orientation to each edge in D, we obtain a diagram D of an oriented spatial trivalent graph. We may represent an orientation of an edge by a normal orientation, which is obtained by rotating a usual orientation $\pi/2$ counterclockwise on the diagram.

For an arc incident to a vertex ω, we define $\epsilon(\alpha;\omega) \in \{1,-1\}$ by

$$\epsilon(\alpha;\omega) = \begin{cases} 1 & \text{the orientation of the arc } \alpha \text{ points to the vertex } \omega, \\ -1 & \text{otherwise.} \end{cases}$$

We denote by $\mathcal{A}(D)$ (resp. $\mathcal{R}(D)$) the set of arcs (resp. complementary regions) of D. Let D be a diagram of an oriented spatial trivalent graph. Let X be a G-family of quandles, Y an X-set, and Q be the associated quandle of X. Let p_X and p_G be the projections from Q to X and G, respectively. An X_Y-*coloring* of D is a map $C : \mathcal{A}(D) \cup \mathcal{R}(D) \to Q \cup Y$ satisfying the following conditions (see Figures 3, 4).

- $C(\mathcal{A}(D)) \subset Q$, $C(\mathcal{R}(D)) \subset Y$.
- Let χ_3 be the over-arc at a crossing χ. Let χ_1, χ_2 be the under-arc at the crossing χ such that the normal orientation of χ_3 points from χ_1 to χ_2. Then
$$C(\chi_2) = C(\chi_1) * C(\chi_3).$$
- Let $\omega_1, \omega_2, \omega_3$ be the arcs incident to a vertex ω. Then
$$(p_X \circ C)(\omega_1) = (p_X \circ C)(\omega_2) = (p_X \circ C)(\omega_3),$$
$$(p_G \circ C)(\omega_1)^{\epsilon(\omega_1;\omega)}(p_G \circ C)(\omega_2)^{\epsilon(\omega_2;\omega)}(p_G \circ C)(\omega_3)^{\epsilon(\omega_3;\omega)} = e.$$
- For any arc $\alpha \in \mathcal{A}(D)$, we have
$$C(\alpha_1) * C(\alpha) = C(\alpha_2),$$
where α_1, α_2 are the regions facing the arc α so that the normal orientation of α points from α_1 to α_2.

We denote by $\mathrm{Col}_X(D)_Y$ the set of X_Y-colorings of D.

For two diagrams D and E which locally differ, we denote by $\mathcal{A}(D,E)$ (resp. $\mathcal{R}(D,E)$) the set of arcs (resp. regions) that D and E share.

Lemma 4. Let X be a G-family of quandles, and Y an X-set. Let D be a diagram of an oriented spatial trivalent graph. Let E be a diagram obtained by applying one of the R1–R6 moves to the diagram D once, where we choose orientations for E which agree with those for D on $\mathcal{A}(D,E)$. For $C \in \mathrm{Col}_X(D)_Y$, there is a unique X_Y-coloring $C_{D,E} \in \mathrm{Col}_X(E)_Y$ such that $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$ and $C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{R}(D,E)}$.

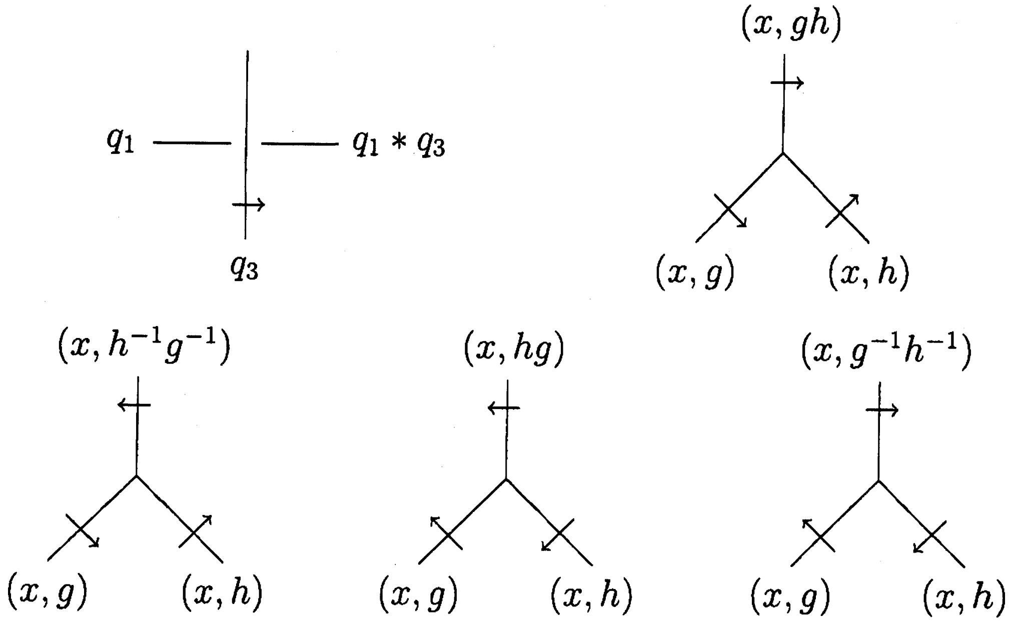

Figure 3:

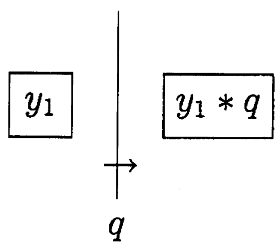

Figure 4:

4 A homology

Let X be a G-family of quandles, Y an X-set, and Q the associated quandle of X. Let $B_n(X)_Y$ be the free abelian group generated by the elements of $Y \times Q^n$ if $n \geq 0$, and let $B_n(X)_Y = 0$ otherwise. We put

$$((y, q_1, \ldots, q_i) * q, q_{i+1}, \ldots, q_n) := (y * q, q_1 * q, \ldots, q_i * q, q_{i+1}, \ldots, q_n)$$

for $y \in Y$ and $q, q_1 \ldots, q_n \in Q$. We define a boundary homomorphism $\partial_n : B_n(X)_Y \to B_{n-1}(X)_Y$ by

$$\begin{aligned}\partial_n(y, q_1, \ldots, q_n) = &\sum_{i=1}^{n}(-1)^i(y, q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n)\\ &-\sum_{i=1}^{n}(-1)^i((y, q_1, \ldots, q_{i-1}) * q_i, q_{i+1}, \ldots, q_n)\end{aligned}$$

for $n > 0$, and $\partial_n = 0$ otherwise. Then $B_*(X)_Y = (B_n(X)_Y, \partial_n)$ is a chain complex (see [1, 2, 3, 4]).

Let $D_n(X)_Y$ be the subgroup of $B_n(X)_Y$ generated by the elements of

$$\bigcup_{i=1}^{n-1}\left\{(y, q_1, \ldots, q_{i-1}, (x, g), (x, h), q_{i+2}, \ldots, q_n) \;\middle|\; \begin{array}{l} y \in Y,\ x \in X,\ g, h \in G \\ q_1, \ldots, q_n \in Q \end{array}\right\}$$

and

$$\bigcup_{i=1}^{n}\left\{\begin{array}{l}(y, q_1, \ldots, q_{i-1}, (x, gh), q_{i+1}, \ldots, q_n) \\ -(y, q_1, \ldots, q_{i-1}, (x, g), q_{i+1}, \ldots, q_n) \\ -((y, q_1, \ldots, q_{i-1}) * (x, g), (x, h), q_{i+1}, \ldots, q_n)\end{array} \;\middle|\; \begin{array}{l} y \in Y,\ x \in X, \\ g, h \in G, \\ q_1, \ldots, q_n \in Q \end{array}\right\}.$$

Lemma 5. For $n \in \mathbb{Z}$, we have $\partial_n(D_n(X)_Y) \subset D_{n-1}(X)_Y$. Thus $D_*(X)_Y = (D_n(X)_Y, \partial_n)$ is a subcomplex of $B_*(X)_Y$.

We put $C_n(X)_Y = B_n(X)_Y/D_n(X)_Y$. Then $C_*(X)_Y = (C_n(X)_Y, \partial_n)$ is a chain complex. For an abelian group A, we define the cochain complex $C^*(X; A)_Y = \mathrm{Hom}(C_*(X)_Y, A)$. We denote by $H_n(X)_Y$ the nth homology group of $C_*(X)_Y$.

5 A cocycle invariant

Let D be a diagram of an oriented spatial trivalent graph. For an X_Y-coloring $C \in \mathrm{Col}_X(D)_Y$, we define the weight $w(\chi; C) \in C_2(X)_Y$ at a crossing χ of D as follows. Let χ_1, χ_2 and χ_3 be respectively the under-arcs and the over-arc at a crossing χ such that the normal orientation of χ_3 points from χ_1 to χ_2. Let R_χ be the region facing χ_1 and χ_3 such that the normal orientations χ_1 and χ_3 point from R_χ to the opposite regions with respect to χ_1 and χ_3, respectively. Then we define

$$w(\chi; C) = \epsilon(\chi)(C(R_\chi), C(\chi_1), C(\chi_3)),$$

where $\epsilon(\chi) \in \{1, -1\}$ is the sign of a crossing χ. We define a chain $W(D; C) \in C_2(X)_Y$ by

$$W(D; C) = \sum_\chi w(\chi; C),$$

where χ runs over all crossings of D.

Lemma 6. The chain $W(D; C)$ is a 2-cycle of $C_*(X)_Y$. Further, for cohomologous 2-cocycles θ, θ' of $C^*(X; A)_Y$, we have $\theta(W(D; C)) = \theta'(W(D; C))$.

Lemma 7. Let D be a diagram of an oriented spatial trivalent graph. Let E be a diagram obtained by applying one of the R1–R6 moves to the diagram D once, where we choose orientations for E which agree with those for D on $\mathcal{A}(D, E)$. For $C \in \mathrm{Col}_X(D)_Y$ and $C_{D,E} \in \mathrm{Col}_X(E)_Y$ such that $C|_{\mathcal{A}(D,E)} = C_{D,E}|_{\mathcal{A}(D,E)}$ and $C|_{\mathcal{R}(D,E)} = C_{D,E}|_{\mathcal{R}(D,E)}$, we have $[W(D; C)] = [W(E; C_{D,E})] \in H_2(X)_Y$.

We denote by G_H (resp. G_K) the fundamental group of the exterior of a handlebody-link H (resp. a spatial graph K). When H is represented by K, G_H and G_K are identical. Let D be a diagram of an oriented spatial trivalent graph K. By the definition of an X_Y-coloring C of D, the map $p_G \circ C|_{\mathcal{A}(D)}$ represents a homomorphism from G_K to G, which we denote by $\rho_C \in \mathrm{Hom}(G_K, G)$. For $\rho \in \mathrm{Hom}(G_K, G)$, we define

$$\mathrm{Col}_X(D; \rho)_Y = \{C \in \mathrm{Col}_X(D)_Y \mid \rho_C = \rho\}.$$

For a 2-cocycle θ of $C^*(X;A)_Y$, we define

$$\begin{aligned}\mathcal{H}(D) &:= \{[W(D;C)] \in H_2(X)_Y \mid C \in \mathrm{Col}_X(D)_Y\},\\ \Phi_\theta(D) &:= \{\theta(W(D;C)) \in A \mid C \in \mathrm{Col}_X(D)_Y\},\\ \mathcal{H}(D;\rho) &:= \{[W(D;C)] \in H_2(X)_Y \mid C \in \mathrm{Col}_X(D;\rho)_Y\},\\ \Phi_\theta(D;\rho) &:= \{\theta(W(D;C)) \in A \mid C \in \mathrm{Col}_X(D;\rho)_Y\}\end{aligned}$$

as multisets.

Lemma 8. Let D be a diagram of an oriented spatial trivalent graph K. For $\rho, \rho' \in \mathrm{Hom}(G_K, G)$ such that ρ and ρ' are conjugate, we have

$$\mathcal{H}(D;\rho) = \mathcal{H}(D;\rho') \qquad \Phi_\theta(D;\rho) = \Phi_\theta(D;\rho').$$

We denote by $\mathrm{Conj}(G_K, G)$ the set of conjugacy classes of homomorphisms from G_K to G. By Lemma 8, $\mathcal{H}(D;\rho)$ and $\Phi_\theta(D;\rho)$ are well-defined for $\rho \in \mathrm{Conj}(G_K, G)$.

Lemma 9. Let D be a diagram of an oriented spatial trivalent graph K. Let E be a diagram obtained from D by reversing the orientation of an edge e. For $\rho \in \mathrm{Hom}(G_K, G)$, we have

$$\begin{aligned}\mathcal{H}(D) &= \mathcal{H}(E), & \mathcal{H}(D;\rho) &= \mathcal{H}(E;\rho),\\ \Phi_\theta(D) &= \Phi_\theta(E), & \Phi_\theta(D;\rho) &= \Phi_\theta(E;\rho).\end{aligned}$$

By Lemma 9, $\mathcal{H}(D)$, $\Phi_\theta(D)$, $\mathcal{H}(D;\rho)$ and $\Phi_\theta(D;\rho)$ are well-defined for a diagram D of an unoriented spatial trivalent graph, which is a diagram of a handlebody-link. For a diagram D of a handlebody-link H, we define

$$\begin{aligned}\mathcal{H}^{\mathrm{hom}}(D) &:= \{\mathcal{H}(D;\rho) \mid \rho \in \mathrm{Hom}(G_H, G)\},\\ \Phi_\theta^{\mathrm{hom}}(D) &:= \{\Phi_\theta(D;\rho) \mid \rho \in \mathrm{Hom}(G_H, G)\},\\ \mathcal{H}^{\mathrm{conj}}(D) &:= \{\mathcal{H}(D;\rho) \mid \rho \in \mathrm{Conj}(G_H, G)\},\\ \Phi_\theta^{\mathrm{conj}}(D) &:= \{\Phi_\theta(D;\rho) \mid \rho \in \mathrm{Conj}(G_H, G)\}\end{aligned}$$

as "multisets of multisets." We remark that, for X_Y-colorings C and $C_{D,E}$ in Lemma 7, we have $\rho_C = \rho_{C_{D,E}}$. Then, by Lemmas 6–9, we have the following theorem.

Theorem 10. Let X be a G-family of quandles, Y an X-set. Let θ be a 2-cocycle of $C^*(X;A)_Y$. Let H be a handlebody-link represented by a diagram D. Then the followings are invariants of a handlebody-link H.

$$\mathcal{D}(H), \quad \Phi_\theta(D), \quad \mathcal{H}^{\mathrm{hom}}(D), \quad \Phi_\theta^{\mathrm{hom}}(D), \quad \mathcal{H}^{\mathrm{conj}}(D), \quad \Phi_\theta^{\mathrm{conj}}(D).$$

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